

# Uniqueness of integrable solutions to $\nabla\zeta = G\zeta$ , $\zeta|_{\Gamma} = 0$ for integrable tensor-coefficients $G$ and applications to elasticity

Johannes Lankeit<sup>1</sup>, Patrizio Neff<sup>1,2</sup>, and Dirk Pauly<sup>1</sup>

<sup>1</sup>Universität Duisburg-Essen, Fakultät für Mathematik

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## Abstract

Let  $\Omega \subset \mathbb{R}^N$  be a Lipschitz domain and  $\Gamma$  be a relatively open and non-empty subset of its boundary  $\partial\Omega$ . We show that the solution to the linear first order system

$$\nabla\zeta = G\zeta, \quad \zeta|_{\Gamma} = 0 \tag{1}$$

vanishes if  $G \in L^1(\Omega; \mathbb{R}^{(N \times N) \times N})$  and  $\zeta \in W^{1,1}(\Omega; \mathbb{R}^N)$ , which is the case e.g. for square integrable solutions  $\zeta$  of (1) and  $G \in L^2(\Omega; \mathbb{R}^{(N \times N) \times N})$ . As a consequence, we prove

$$\|\cdot\| : C_0^\infty(\Omega, \Gamma; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)}$$

to be a norm for  $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  with  $\text{Curl } P \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ ,  $\text{Curl } P^{-1} \in L^q(\Omega; \mathbb{R}^{3 \times 3})$  for some  $p, q > 1$  with  $1/p + 1/q = 1$  as well as  $\det P \geq c^+ > 0$ . We also give a new and different proof for the so called ‘infinitesimal rigid displacement lemma’ in curvilinear coordinates: Let  $\Phi \in H^1(\Omega; \mathbb{R}^3)$  satisfy  $\text{sym}(\nabla\Phi^\top \nabla\Psi) = 0$  for some  $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap H^2(\Omega; \mathbb{R}^3)$  with  $\det \nabla\Psi \geq c^+ > 0$ . Then there exists a constant translation vector  $a \in \mathbb{R}^3$  and a constant skew-symmetric matrix  $A \in \mathfrak{so}(3)$ , such that  $\Phi = A\Psi + a$ .

**Key words:** Korn’s inequality, generalized Korn’s first inequality, first order system of partial differential equations, uniqueness, infinitesimal rigid displacement lemma, Korn’s inequality in curvilinear coordinates, unique continuation

## 1 Introduction

Consider the linear first order system of partial differential equations

$$\nabla\zeta = G\zeta, \quad \zeta|_{\Gamma} = 0. \tag{2}$$

Obviously, one solution is  $\zeta = 0$ . But is this solution unique? The answer is not as obvious as it may seem; consider for example in dimension  $N := 1$ ,  $G(t) := 1/t$  in the domain  $\Omega := (0, 1)$  with  $\Gamma := \{0\} \subset \partial\Omega$ . Then  $\zeta := \text{id} \neq 0$  solves (2). However, in the latter example the solution becomes unique if  $G \in L^1(\Omega)$ , which is easily deduced from Gronwall’s lemma. Here we can see that we will need integrability conditions on the coefficient  $G$ ; for a precise formulation of the result see section 2. The uniqueness of the solution to (2) makes

$$\|u\| := \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)} \tag{3}$$

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<sup>2</sup>To whom correspondence should be addressed. e-mail: patrizio.neff@uni-due.de

a norm on

$$\begin{aligned} C_o^\infty(\Omega, \Gamma; \mathbb{R}^3) &:= \{u \in C^\infty(\overline{\Omega}; \mathbb{R}^3) : \text{dist}(\text{supp } u, \Gamma) > 0\}, \\ C^\infty(\overline{\Omega}; \mathbb{R}^3) &:= \{u|_\Omega : u \in C_o^\infty(\mathbb{R}^3; \mathbb{R}^3)\} \end{aligned}$$

for  $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  with  $\det P \geq c^+ > 0$  if  $\text{Curl } P \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ ,  $\text{Curl } P^{-1} \in L^q(\Omega; \mathbb{R}^{3 \times 3})$  for some  $p, q > 1$  and  $1/q + 1/p = 1$ . Here the Curl of a matrix field is defined as the row-wise standard curl in  $\mathbb{R}^3$ .

The question whether an expression of the form (3) is a norm arises when trying to generalize Korn's first inequality to hold for non-constant coefficients, i.e.,

$$\exists c > 0 \quad \forall u \in H_o^1(\Omega, \Gamma; \mathbb{R}^3) \quad \| \text{sym}(\nabla u P^{-1}) \|_{L^2(\Omega)} \geq c \|u\|_{H^1(\Omega)}, \quad (4)$$

which was first done for  $P, P^{-1}, \text{Curl } P \in C^1(\overline{\Omega}; \mathbb{R}^{3 \times 3})$  by Neff in [7], cf. [17]. Here  $H_o^1(\Omega, \Gamma; \mathbb{R}^3)$  denotes the closure of  $C_o^\infty(\Omega, \Gamma; \mathbb{R}^3)$  in  $H^1(\Omega; \mathbb{R}^3)$ . The classical Korn's first inequality is obtained for  $P$  being the identity matrix, see [6, 3, 7, 13, 14, 15]. The inequality (4) has been proved in [17] to hold for continuous  $P^{-1}$ , whereas it can be violated for  $P^{-1} \in L^\infty(\Omega)$  or  $P^{-1} \in SO(3)$  a.e.. The counterexamples, given by Pompe in [17] and [18], see also [16], each use the fact that for such  $P$  an expression of the form of  $\|\cdot\|$  is not a norm (It has a nontrivial kernel) on the spaces of functions considered. Quadratic forms of the type (4) arise in applications to geometrically exact models of shells, plates and membranes, in micromorphic and Cosserat type models and in plasticity, [5, 8, 10, 11, 9].

The so called ‘infinitesimal rigid displacement lemma in curvilinear coordinates’, a version of which can be found in [1] and which is important for linear elasticity in curvilinear coordinates (see also [2, 4]) states the following: If  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}^N)$  satisfying  $\det \nabla \Psi \geq c^+ > 0$  a.e. and  $\Phi \in H^1(\Omega; \mathbb{R}^N)$  with  $\text{sym}(\nabla \Phi^\top \nabla \Psi) = 0$  a.e., then on a dense open subset of  $\Omega$  there exist locally constant mappings  $a : \Omega \rightarrow \mathbb{R}^N$  and  $A : \Omega \rightarrow \mathfrak{so}(N)$  such that locally  $\Phi = A\Psi + a$ . If  $\Omega$  is Lipschitz then the terms ‘locally’ can be dropped. In their proof [1], the authors apply the chain rule to  $\Theta := \Phi \circ \Psi^{-1}$  and use the observation that the conditions  $\text{sym}(\nabla \Phi^\top \nabla \Psi) = 0$  and  $\text{sym}(\nabla \Phi(\nabla \Psi)^{-1}) = 0$  are equivalent by a clever conjugation with  $(\nabla \Psi)^{-1}$ , this is

$$(\nabla \Psi)^{-\top} \text{sym}(\nabla \Phi^\top \nabla \Psi)(\nabla \Psi)^{-1} = \text{sym}(\nabla \Phi(\nabla \Psi)^{-1}) = \text{sym}(\nabla(\Phi \circ \Psi^{-1})) \circ \Psi, \quad (5)$$

together with the classical infinitesimal rigid displacement lemma applied on  $\Theta$ , defined on the domain  $\Psi(\Omega)$ . If to this lemma a boundary condition  $\Phi = 0$  on a relatively open subset of the boundary is added, one obtains  $\Phi = 0$  (cf. [2, 1.7-3(b)]).

The main part of our proof for  $\|\cdot\|$  being a norm is also concerned with obtaining  $u = 0$  from  $\text{sym}(\nabla u P^{-1}) = 0$ . By taking  $P = \nabla \Psi$  to be a gradient, we present another proof of the infinitesimal rigid displacement lemma in dimension  $N = 3$  which yields  $\Phi = A\Psi + a$  with  $A \in \mathfrak{so}(N)$ ,  $a \in \mathbb{R}^N$ . We need slightly more regularity but do not use the chain rule for  $\Theta$ .

The key tool for obtaining our results is Neff's formula for the Curl of the product of two matrices, the first of which is skew-symmetric. We state a generalization of this formula in section 4.1.

This paper is organized as follows: The next section states the main results that will be proven in the subsequent chapters. Section 3.1 provides a tool that gives  $\zeta = 0$  on lines and is used in section 3.2 where this is extended to cubes. Section 3.3 then takes care of the whole domain if a ‘( $\zeta = 0$ )-cube’ is given as starting point. In section 3.4 the uniqueness theorem of section 2 is proven, mainly by putting together the results of the previous sections. After that and before applying the theorem we have a closer look at the formula for the Curl of a product of matrices (section 4.1). Finally, in sections 4.2 and 5, respectively, we prove that  $\|\cdot\|$  is a norm and we present our new proof of the infinitesimal rigid displacement lemma.

## 2 Results

Let us first note that by  $\nabla$  we denote not only the gradient of a scalar-valued function, but also (as an usual gradient row-wise) the derivative or Jacobian of a vector-field. The Curl of a matrix is to be taken row-wise as usual curl for vector fields.

**Theorem 2.1 (Unique Continuation).** *Let  $\Omega \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ , be a Lipschitz domain,  $\Gamma$  be a relatively open and non-empty subset of  $\partial\Omega$  as well as  $G \in L^1(\Omega; \mathbb{R}^{(N \times N) \times N})$ . If  $\zeta \in W^{1,1}(\Omega; \mathbb{R}^N)$  solves*

$$\nabla \zeta = G \zeta, \quad \zeta|_{\Gamma} = 0,$$

*then  $\zeta = 0$ .*

From the differential equation itself it is not a priori clear that  $\zeta$  belongs to  $W^{1,1}(\Omega)$ . But this can be ensured by requiring higher integrability of  $G$  and  $\zeta$ , since for bounded domains, e.g., the conditions  $G \in L^2(\Omega)$  and  $\zeta \in L^2(\Omega)$  imply  $\nabla \zeta \in L^1(\Omega)$  and hence  $\zeta \in W^{1,1}(\Omega)$ , where an application of the theorem ensures  $\zeta = 0$ . Thus we have obtained the uniqueness of  $L^2(\Omega)$ -solutions if the coefficient  $G$  are square-integrable. Of course, the same holds if  $\zeta \in L^p(\Omega)$  for arbitrary  $p \geq 1$ . Then  $G$  at least needs to be an  $L^q(\Omega)$ -function, where  $1/p + 1/q = 1$ .

**Theorem 2.2 (Norm).** *Let  $\Omega \subset \mathbb{R}^3$  be a Lipschitz domain,  $\emptyset \neq \Gamma \subset \partial\Omega$  be relatively open,  $P \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$  with  $\det P \geq c^+ > 0$ ,  $\text{Curl } P \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ ,  $\text{Curl } P^{-1} \in L^q(\Omega; \mathbb{R}^{3 \times 3})$  for some  $p, q > 1$  with  $1/p + 1/q = 1$ . Then*

$$\|\cdot\| : C_o^\infty(\Omega, \Gamma; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \|\text{sym}(\nabla u P^{-1})\|_{L^2(\Omega)} \quad (6)$$

*defines a norm.*

**Remark 2.3.** *In the case of  $p = q = 2$  and for  $P \in SO(3)$  a.e.,  $\text{Curl } P^{-1} \in L^2(\Omega)$  is no additional condition, since then  $\text{Curl } P \in L^2(\Omega) \Leftrightarrow \text{Curl } P^{-1} \in L^2(\Omega)$ . (Note that for  $P \in SO(3)$  a.e. generally  $P, \text{Curl } P \in L^p(\Omega)$  is equivalent to  $P \in W^{1,p}(\Omega)$ , cf. [12].)*

**Conjecture 2.4.** *Theorem 2.2 holds for  $P \in L^\infty(\Omega)$  with  $\text{Curl } P \in L^p(\Omega)$  and  $\det P \geq c^+ > 0$  for some  $p > 1$  or even  $p \geq 1$ .*

**Remark 2.5.** *Since the norms  $\|\cdot\|$  and  $\|\cdot\|_{H^1(\Omega)}$  are not shown to be equivalent, it is not clear, whether the spaces  $H_o^1(\Omega, \Gamma) = \overline{C_o^\infty(\Omega, \Gamma)}^{\|\cdot\|_{H^1(\Omega)}}$  and  $\overline{C_o^\infty(\Omega, \Gamma)}^{\|\cdot\|}$  coincide. However, by [17], these norms are equivalent if  $P \in C^0(\overline{\Omega})$  with  $\det P \geq c^+ > 0$ .*

**Conjecture 2.6.** *The norms are equivalent if  $P \in L^\infty(\Omega)$  with  $\text{Curl } P \in L^p(\Omega)$  and  $\det P \geq c^+ > 0$  for some  $p > 1$  or even  $p \geq 1$ .*

**Theorem 2.7 (Infinitesimal Rigid Displacement Lemma).** *Let  $\Omega \subset \mathbb{R}^3$  be a Lipschitz domain. Moreover, let  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^3)$  and  $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}^3) \cap W^{2,q}(\Omega; \mathbb{R}^3)$  with  $\det \nabla \Psi \geq c^+ > 0$  a.e. and  $p, q > 1$ ,  $1/p + 1/q = 1$ . If*

$$\text{sym}(\nabla \Phi^\top \nabla \Psi) = 0$$

*then there exist  $a \in \mathbb{R}^3$  and a constant skew-symmetric matrix  $A \in \mathfrak{so}(3)$ , such that  $\Phi = A\Psi + a$ .*

**Remark 2.8.** *When comparing two nearby configurations of an elastic body, namely  $\hat{\Psi} : \Omega \rightarrow \mathbb{R}^3$  and  $\Psi : \Omega \rightarrow \mathbb{R}^3$ , following Ciarlet [4] we may always write  $\hat{\Psi} = \Psi + \Phi$ , where  $\Phi : \Omega \rightarrow \mathbb{R}^3$  is the displacement from  $\Psi$  to  $\hat{\Psi}$ . The respective metric tensors of the two configurations are  $\nabla \hat{\Psi}^\top \nabla \hat{\Psi}$  and  $\nabla \Psi^\top \nabla \Psi$ . In terms of the displacement  $\Phi$  to lowest order we have for the  $\Phi$ -linearized change of the metric*

$$\begin{aligned} [\nabla \hat{\Psi}^\top \nabla \hat{\Psi} - \nabla \Psi^\top \nabla \Psi]_{\text{lin}, \Phi} &= [(\nabla \Psi + \nabla \Phi)^\top (\nabla \Psi + \nabla \Phi) - \nabla \Psi^\top \nabla \Psi]_{\text{lin}, \Phi} \\ &= \nabla \Phi^\top \nabla \Psi + \nabla \Psi^\top \nabla \Phi = 2 \text{sym}(\nabla \Phi^\top \nabla \Psi). \end{aligned}$$

*Therefore, the infinitesimal rigid displacement lemma expresses the fact that if the linearized change of the metric is zero, then the displacement must be (the linearized part of) some rigid displacement.*

### 3 Proof of the uniqueness theorem

We start with some preliminaries.

#### 3.1 Vanishing in intervals

Let  $-\infty < a < b < \infty$  and  $I := (a, b)$ .

**Lemma 3.1.** *Let  $G \in L^1(I; \mathbb{R}^{N \times N})$ ,  $\zeta \in W^{1,1}(I; \mathbb{R}^N)$  with  $\zeta' = G\zeta$  and  $\zeta(a) = 0$ . Then  $\zeta = 0$ .*

*Proof.* We use Gronwall's inequality. Because  $\zeta \in W^{1,1}(I)$ ,  $\zeta$  is absolutely continuous and hence it can be written as an integral over its derivative:

$$\zeta(x) = \zeta(a) + \int_a^x \zeta'(t) dt = \zeta(a) + \int_a^x G(t)\zeta(t) dt \Rightarrow |\zeta(x)| \leq |\zeta(a)| + \int_a^x \|G(t)\| |\zeta(t)| dt.$$

An application of Gronwall's inequality leads to

$$|\zeta(x)| \leq |\zeta(a)| \exp\left(\int_a^x \|G(t)\| dt\right) = 0, \quad x \in I,$$

which concludes the proof.  $\square$

#### 3.2 Vanishing in cubes

Let  $Q$  be a cuboid in  $\mathbb{R}^N$  and let  $\Gamma$  be a face of  $Q$ , i.e.,  $Q = \Gamma \times I$  from the previous section. By [19, Th. 2.1.4] we have that for  $u \in L^1(Q)$  the following is equivalent:  $u \in W^{1,1}(Q)$ , if and only if  $u$  has a representative which is absolutely continuous on almost all line segments in  $Q$  parallel to the coordinate axes and whose (classical a.e.) partial derivatives belong to  $L^1(Q)$ . These classical partial derivatives coincide with the weak derivatives almost everywhere. In particular, if  $u \in W^{1,1}(Q)$  then  $u_\gamma := u(\gamma, \cdot) \in W^{1,1}(I)$  f.a.a.  $\gamma \in \Gamma$ . Of course, the same holds for  $u \in W^{1,p}(Q)$  with  $p \geq 1$ .

**Lemma 3.2.** *Let  $G \in L^1(Q; \mathbb{R}^{(N \times N) \times N})$  and  $\zeta \in W^{1,1}(Q; \mathbb{R}^N)$  with  $\nabla\zeta = G\zeta$  and  $\zeta|_\Gamma = 0$ . Then  $\zeta = 0$ .*

*Proof.* Since

$$\|\zeta\|_{L^1(\Omega)} = \int_\Gamma \int_a^b |\zeta(\gamma, x)| dx d\gamma = \int_\Gamma \int_a^b |\zeta_\gamma(x)| dx d\gamma$$

we only have to show  $\zeta_\gamma = 0$  a.e.. As  $\zeta \in W^{1,1}(Q)$ ,  $\zeta_\gamma \in W^{1,1}(I)$  f.a.a.  $\gamma \in \Gamma$  by [19, Th. 2.1.4], as mentioned before. Since  $\zeta'_\gamma$  is the last column  $\nabla\zeta e^N$  of  $\nabla\zeta$ , we have

$$\zeta'_\gamma = \nabla\zeta(\gamma, \cdot)e^N = G(\gamma, \cdot)\zeta(\gamma, \cdot)e^N = G(\gamma, \cdot)\zeta_\gamma e^N =: G_\gamma \zeta_\gamma.$$

For fixed  $(\gamma, x) \in Q$ ,  $G(\gamma, x)$  is a linear mapping from  $\mathbb{R}^N$  to  $\mathbb{R}^{N \times N}$ , its product with  $\zeta_\gamma(x) \in \mathbb{R}^N$  is an element of  $\mathbb{R}^{N \times N}$  and multiplication by  $e^N$  gives an element of  $\mathbb{R}^N$  depending linearly on  $\zeta_\gamma(x)$ . Hence,  $G_\gamma(x)$  is a linear mapping from  $\mathbb{R}^N$  to  $\mathbb{R}^N$  a.e.. Even  $G_\gamma \in L^1(I; \mathbb{R}^{N \times N})$  holds, since  $G \in L^1(Q)$ . Also,  $\zeta|_\Gamma = 0$  implies  $\zeta_\gamma(a) = 0$  f.a.a.  $\gamma \in \Gamma$ . By Lemma 3.1 we obtain  $\zeta_\gamma = 0$ .  $\square$

#### 3.3 Unique continuation

**Lemma 3.3.** *Let  $G \in L^1(\Omega; \mathbb{R}^{(N \times N) \times N})$  and  $\zeta \in W^{1,1}(\Omega; \mathbb{R}^N)$  with  $\nabla\zeta = G\zeta$ . Moreover, let  $\zeta$  vanish in an open ball  $B \subset \Omega$ . Then  $\zeta = 0$ .*

*Proof.* Let  $\Omega$  be convex and pick some  $x_1 \in B$ . Then we can take a straight line between  $x_1$  and some other point  $x_2 \in \Omega$  and a cuboid  $Q$  containing this line and having one face being entirely located in  $B$ . Then by Lemma 3.2  $\zeta = 0$  in  $Q$  and hence in a whole neighborhood of  $x_2$ . Since  $x_2$  was arbitrary, we have  $\zeta = 0$  in  $\Omega$ . By induction this can be carried over to connected unions of finitely many convex sets and hence works for path-connected sets, because every path between two points can be covered by such a finite union. Since domains are path-connected, we finally achieve  $\zeta = 0$  in  $\Omega$ .  $\square$

**Remark 3.4.** Lemma 3.3 can also be stated as: The equation  $\nabla\zeta = G\zeta$ , i.e., the operator  $\nabla - G$  has the unique continuation property. Moreover, it is enough that  $\zeta$  vanishes on a small part of some  $(N-1)$ -dimensional hyper-plane.

### 3.4 Proof of Theorem 2.1

Let  $\zeta$  be as in Theorem 2.1. If we can show that  $\zeta$  vanishes on an open set, we can apply Lemma 3.3 and hence,  $\zeta$  must vanish in the whole of  $\Omega$ . To make  $\zeta = 0$  on an open set, we transform a part of  $\Gamma$ , where we know  $\zeta$  to be zero, and a neighborhood  $U$  onto a cuboid  $Q$ , where we can use Lemma 3.2 and the transformed function is forced to vanish, hence also  $\zeta$  must vanish on  $U$ . Let us pick a point on  $\Gamma$  and a corresponding open neighborhood  $U$  as well as a bijective bi-Lipschitz transformation  $\varphi : \tilde{Q} := (-1, 1)^N \rightarrow U$ , mapping the cuboid  $Q := (-1, 1)^{N-1} \times (0, 1)$  onto  $U \cap \Omega$  and  $(-1, 1)^{N-1} \times \{0\}$  onto  $U \cap \Gamma$ . Now  $\tilde{G} := G \circ \varphi \in L^1(Q; \mathbb{R}^{(N \times N) \times N})$  and, see again e.g. [19, Th. 2.2.2],  $\tilde{\zeta} := \zeta \circ \varphi \in W^{1,1}(Q; \mathbb{R}^N)$ . By the chain rule we have

$$\nabla \tilde{\zeta} = ((\nabla \zeta) \circ \varphi) \nabla \varphi = ((G \zeta) \circ \varphi) \nabla \varphi = \tilde{G} \tilde{\zeta} \nabla \varphi =: \hat{G} \tilde{\zeta}.$$

Since  $\nabla \varphi$  is uniformly bounded we get

$$\forall z \in Q, y \in \mathbb{R}^N \quad \|\hat{G}(z)y\| \leq \|\tilde{G}(z)\| |y| \|\nabla \varphi(z)\| \leq c \|\tilde{G}(z)\| |y| \quad \Rightarrow \quad \|\hat{G}(z)\| \leq c \|\tilde{G}(z)\|$$

and hence

$$\int_Q \|\hat{G}(z)\| dz \leq c \int_Q \|\tilde{G}(z)\| dz = c \int_{\varphi(Q)} \|G(x)\| |\det \nabla \varphi^{-1}(x)| dx \leq c \int_{\Omega \cap U} \|G(x)\| dx < \infty$$

since  $\det \nabla \varphi^{-1}$  is uniformly bounded as well. Thus  $\hat{G} \in L^1(Q; \mathbb{R}^{(N \times N) \times N})$ . Because  $\zeta$  vanishes on  $U \cap \Gamma$ ,  $\tilde{\zeta}$  vanishes on  $F := \varphi^{-1}(U \cap \Gamma) = (-1, 1)^{N-1} \times \{0\}$ . Hence  $\tilde{\zeta} \in W^{1,1}(Q; \mathbb{R}^N)$  solves

$$\nabla \tilde{\zeta} = \hat{G} \tilde{\zeta}, \quad \tilde{\zeta}|_F = 0.$$

Lemma 3.2 implies  $\tilde{\zeta} = 0$  in  $Q$ . Thus  $\zeta = \tilde{\zeta} \circ \varphi^{-1} = 0$  in  $U \cap \Omega$ , which contains an open ball.  $\square$

## 4 Proof of the norm property

### 4.1 Curl of matrix-products

We identify  $\mathbb{R}^{3 \times 3}$  and  $\mathbb{R}^9$  by the following isomorphisms:

$$\text{mat} : \mathbb{R}^9 \rightarrow \mathbb{R}^{3 \times 3}, \quad \begin{bmatrix} a_1 \\ \vdots \\ a_9 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \quad \text{vec} := \text{mat}^{-1} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^9$$

We also use the following canonical isomorphism to identify  $\mathbb{R}^3$  and  $\mathfrak{so}(3)$ :

$$\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3, \quad \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Moreover, we define

$$\begin{aligned} \text{diagvec} : \mathbb{R}^{3 \times 3} &\rightarrow \mathbb{R}^3, & \begin{bmatrix} \textcircled{1} & a_2 & a_3 \\ a_4 & \textcircled{5} & a_6 \\ a_7 & a_8 & \textcircled{9} \end{bmatrix} &\mapsto \begin{bmatrix} a_1 \\ a_5 \\ a_9 \end{bmatrix}, \\ \text{skewvec} : \mathbb{R}^{3 \times 3} &\rightarrow \mathbb{R}^3, & \begin{bmatrix} a_1 & \textcircled{2} & \textcircled{3} \\ a_4 & a_5 & \textcircled{6} \\ a_7 & a_8 & a_9 \end{bmatrix} &\mapsto \begin{bmatrix} -a_6 \\ a_3 \\ -a_2 \end{bmatrix}, \\ \text{symvec} : \mathbb{R}^{3 \times 3} &\rightarrow \mathbb{R}^3, & \begin{bmatrix} a_1 & a_2 & a_3 \\ \textcircled{4} & a_5 & a_6 \\ \textcircled{7} & \textcircled{8} & a_9 \end{bmatrix} &\mapsto \begin{bmatrix} a_8 \\ -a_7 \\ a_4 \end{bmatrix}. \end{aligned}$$

We note  $\text{skewvec} = \text{symvec} = \text{axl}$ ,  $\text{diagvec} = 0$  on  $\mathfrak{so}(3)$ . Furthermore,  $Ax = \text{axl}(A) \times x$  and  $\text{axl}^{-1}(a)x = a \times x$  holds for all  $A \in \mathfrak{so}(3)$  and all  $a, x \in \mathbb{R}^3$ , where  $\times$  denotes the cross-product.

For a matrix  $Y \in \mathbb{R}^{3 \times 3}$  with  $Y^\top = [y_1 \ y_2 \ y_3]$  and vectors  $y_n \in \mathbb{R}^3$  we define

$$\begin{aligned} L_{\text{diag}, Y} &= - \begin{bmatrix} \text{axl}^{-1} y_1 & 0 & 0 \\ 0 & \text{axl}^{-1} y_2 & 0 \\ 0 & 0 & \text{axl}^{-1} y_3 \end{bmatrix}, \\ L_{\text{skew}, Y} &= \begin{bmatrix} 0 & -\text{axl}^{-1} y_3 & \text{axl}^{-1} y_2 \\ \text{axl}^{-1} y_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ L_{\text{sym}, Y} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\text{axl}^{-1} y_1 \\ -\text{axl}^{-1} y_2 & \text{axl}^{-1} y_1 & 0 \end{bmatrix}, \\ L_Y := L_{\text{skew}, Y} + L_{\text{sym}, Y} &= \begin{bmatrix} 0 & -\text{axl}^{-1} y_3 & \text{axl}^{-1} y_2 \\ \text{axl}^{-1} y_3 & 0 & -\text{axl}^{-1} y_1 \\ -\text{axl}^{-1} y_2 & \text{axl}^{-1} y_1 & 0 \end{bmatrix} \end{aligned}$$

and note  $L_Y^\top = L_Y$ . Furthermore, for vector fields  $v$  in  $\mathbb{R}^3$  we set

$$\hat{\nabla} v := \text{vec } \nabla v,$$

denoting the vector-field containing the nine partial derivatives of the three components of  $v$ .

Now, we extend Neff's formula from [7, Lemma 3.7] in two ways, such that it can be applied with weaker differentiability and for general matrices. For this, we define

$$W^s(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3}) := \{Y \in L^s(\Omega; \mathbb{R}^{3 \times 3}) : \text{Curl } Y \in L^s(\Omega; \mathbb{R}^{3 \times 3})\}.$$

**Remark 4.1.** For skew-symmetric matrix fields we have  $W^s(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3}) = W^{1,s}(\Omega; \mathbb{R}^{3 \times 3})$ , since in this case the Curl controls all the derivatives, see [12].

**Lemma 4.2.** Let  $r, s \in (1, \infty)$  with  $1/r + 1/s = 1$ . Moreover, let  $X \in W^{1,r}(\Omega; \mathbb{R}^{3 \times 3})$  and  $Y \in W^s(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3})$ . Then  $XY \in W^1(\text{Curl}, \Omega; \mathbb{R}^{3 \times 3})$  and

$$\text{Curl}(XY) = \text{mat}(L_{\text{diag}, Y} \hat{\nabla} \text{diagvec} + L_{\text{skew}, Y} \hat{\nabla} \text{skewvec} + L_{\text{sym}, Y} \hat{\nabla} \text{symvec})X + X \text{Curl } Y \quad (7)$$

with  $L_{\text{diag}, Y}, L_{\text{skew}, Y}, L_{\text{sym}, Y} \in L^s(\Omega; \mathbb{R}^{9 \times 9})$ . For skew-symmetric  $X$  formula (7) turns to

$$\text{Curl}(XY) = \text{mat } L_Y(\hat{\nabla} \text{axl } X) + X \text{Curl } Y, \quad (8)$$

where  $\det L_Y = -2(\det Y)^3$ . Hence, if  $Y$  is invertible, so is  $L_Y$ .

We note that for smooth ( $C^1$ ) matrices  $X, Y$ , where  $X$  is skew-symmetric, formula (8) was already shown in [7, Lemma 3.7].

*Proof.* Since  $C_c^\infty(\bar{\Omega})$  is dense in both  $W^{1,r}(\Omega)$  and  $W^s(\text{Curl}, \Omega)$  we have to show (7) only for smooth matrix fields. But this is a straight forward calculation, which we present in the appendix. (8) is a simple consequence from (7) and the assertion about the determinants has been proved already in [7, Lemma 3.7].  $\square$

## 4.2 Proof of Theorem 2.2

Let  $u \in C_c^\infty(\Omega, \Gamma; \mathbb{R}^3)$  with  $\|u\| = 0$ . We have to show  $u = 0$ . Note that  $\text{sym}(\nabla u P^{-1}) = 0$  implies

$$\nabla u P^{-1} = A, \quad (9)$$

where  $A$  is some skew-symmetric matrix field. Moreover, since  $AP = \nabla u$  we have

$$\text{Curl}(AP) = 0. \quad (10)$$

Without loss of generality we assume  $\Gamma$  to be bounded (otherwise, replace  $\Gamma$  by a bounded open subset of itself) and that the compact set  $\text{supp } u$  and  $\Gamma$  are both contained in some open ball  $B$ . Define  $\tilde{\Omega} := \Omega \cap B$ . Then  $\nabla u, P, P^{-1}$  and  $A$  belong to  $L^\infty(\tilde{\Omega}, \mathbb{R}^{3 \times 3}) \subset L^r(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$  for all  $r \in [1, \infty]$ .

Since  $\text{Curl } P^{-1} \in L^q(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$ , Lemma 4.2 with  $X := \nabla u$  and  $Y := P^{-1}$  together with Remark 4.1 show  $A \in W^{1,1}(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$  and by (7) even  $A \in W^{1,q}(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$  holds. Another application of Lemma 4.2 with  $X := A$  and  $Y := P$  gives by (8) and (10)

$$\text{mat } L_P(\hat{\nabla} \text{axl } A) + A \text{Curl } P = 0,$$

since  $A$  is skew-symmetric. Thus,  $\zeta := \text{axl } A \in W^{1,q}(\tilde{\Omega}, \mathbb{R}^3) \subset W^{1,1}(\tilde{\Omega}, \mathbb{R}^3)$  solves

$$\nabla \zeta = -\text{mat } L_P^{-1} \text{vec}(\text{axl}^{-1} \zeta \text{Curl } P) =: G_P \zeta. \quad (11)$$

Since  $L_P, L_P^{-1} \in L^\infty(\tilde{\Omega}, \mathbb{R}^{9 \times 9})$  and  $\text{Curl } P \in L^p(\tilde{\Omega}, \mathbb{R}^{3 \times 3}) \subset L^1(\tilde{\Omega}, \mathbb{R}^{3 \times 3})$ , also  $G_P$  belongs to  $L^p(\tilde{\Omega}, \mathbb{R}^{(3 \times 3) \times 3}) \subset L^1(\tilde{\Omega}, \mathbb{R}^{(3 \times 3) \times 3})$ . Additionally,  $A$  and hence  $\zeta$  vanish on  $\Gamma$  by (9) since  $u$  does. By Theorem 2.1,  $\zeta$  and therefore  $A$  and  $\nabla u$  vanish in  $\tilde{\Omega}$ . Thus  $u = 0$  in  $\tilde{\Omega}$  because  $u$  vanishes on  $\Gamma$ . Since  $\text{supp } u \subset \tilde{\Omega}$  we finally obtain  $u = 0$  in  $\Omega$ .  $\square$

## 5 Proof of Theorem 2.7

$A := \nabla \Phi(\nabla \Psi)^{-1} \in L^p(\Omega; \mathbb{R}^{3 \times 3})$  is skew-symmetric by (5). Since the standard mollification preserves skew-symmetry we can pick a sequence  $(A_n) \subset C_c^\infty(\Omega; \mathbb{R}^{3 \times 3})$  of skew-symmetric smooth matrices approximating  $A$  in  $L^p(\Omega)$ . Applying Lemma 4.2, i.e., (8), to  $A_n \nabla \Psi$  we get

$$\text{Curl}(A_n \nabla \Psi) = \text{mat } L_{\nabla \Psi}(\hat{\nabla} \text{axl } A_n)$$

with invertible  $L_{\nabla \Psi} \in W^{1,q}(\Omega)$  satisfying  $L_{\nabla \Psi}^{-1} \in W^{1,q}(\Omega)$  by assumption on the regularity of  $\Psi$ . Pick  $\Theta \in C_c^\infty(\Omega; \mathbb{R}^{3 \times 3})$ . Then  $L_{\nabla \Psi}^{-\top} \text{vec } \Theta \in W_o^{1,q}(\Omega)$  and since  $A \nabla \Psi = \nabla \Phi \in L^p(\Omega; \mathbb{R}^{3 \times 3})$  with  $\text{Curl}(A \nabla \Psi) = 0$  we have

$$\langle A_n \nabla \Psi, \text{Curl}(\text{mat } L_{\nabla \Psi}^{-\top} \text{vec } \Theta) \rangle_{L^2(\Omega)} \rightarrow \langle A \nabla \Psi, \text{Curl}(\text{mat } L_{\nabla \Psi}^{-\top} \text{vec } \Theta) \rangle_{L^2(\Omega)} = 0.$$

On the other hand we have for the left hand side

$$\begin{aligned} \langle A_n \nabla \Psi, \text{Curl}(\text{mat } L_{\nabla \Psi}^{-\top} \text{vec } \Theta) \rangle_{L^2(\Omega)} &= \langle \text{Curl}(A_n \nabla \Psi), \text{mat } L_{\nabla \Psi}^{-\top} \text{vec } \Theta \rangle_{L^2(\Omega)} \\ &= \langle L_{\nabla \Psi}(\hat{\nabla} \text{axl } A_n), L_{\nabla \Psi}^{-\top} \text{vec } \Theta \rangle_{L^2(\Omega)} \\ &= \langle \hat{\nabla} \text{axl } A_n, L_{\nabla \Psi}^\top L_{\nabla \Psi}^{-\top} \text{vec } \Theta \rangle_{L^2(\Omega)} \\ &= \langle \nabla \text{axl } A_n, \Theta \rangle_{L^2(\Omega)} \\ &= \langle \text{axl } A_n, \text{Div } \Theta \rangle_{L^2(\Omega)} \rightarrow \langle \text{axl } A, \text{Div } \Theta \rangle_{L^2(\Omega)}. \end{aligned}$$

Hence,  $\nabla \text{axl } A = 0$  and therefore  $A \in \mathfrak{so}(3)$  is constant. Thus,  $\nabla(\Phi - A\Psi) = \nabla\Phi - A\nabla\Psi = 0$  and  $\Phi = A\Psi + a$  with some  $a \in \mathbb{R}^3$ .  $\square$

## A Appendix

We show (7) for smooth matrix fields  $X = [x_{nm}]_{n,m=1,2,3}$  and  $Y = [y_{nm}]_{n,m=1,2,3}$ . The  $l$ -th row of  $XY$  is the transpose of the vector having the entries  $x_{ln}y_{nk}$  for  $k = 1, 2, 3$ . Thus, the  $l$ -th row of  $\text{Curl}(XY)$  is the transpose of the vector having the entries  $\partial_i(x_{ln}y_{nj}) - \partial_j(x_{ln}y_{ni})$  for  $k = 1, 2, 3$ , where the curl of a vector field  $v$  is written as

$$\text{curl } v = \begin{bmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{bmatrix} = [\partial_i v_j - \partial_j v_i]_{k=1,2,3}.$$

Therefore,

$$\begin{aligned} [\text{Curl}(XY)]_{lk} &= \partial_i x_{ln} y_{nj} - \partial_j x_{ln} y_{ni} + x_{ln} (\underbrace{\partial_i y_{nj} - \partial_j y_{ni}}_{= [\text{Curl } Y]_{nk}}) \\ &= \partial_i x_{ln} y_{nj} - \partial_j x_{ln} y_{ni} + [X \text{Curl } Y]_{lk}. \end{aligned}$$

With the transpose of the  $n$ -th row of  $Y$  denoted by  $[y_n]_j := y_{nj}$ , we get

$$[\text{Curl}(XY)]_{lk} - [X \text{Curl } Y]_{lk} = [\nabla x_{ln} \times y_n]_k$$

and hence for the  $l$ -th row  $[\text{Curl}(XY) - X \text{Curl } Y]_l = [(\nabla x_{ln} \times y_n)^\top]_l$ . Finally we obtain:

$$\begin{aligned} \text{Curl}(XY) - X \text{Curl } Y &= \begin{bmatrix} (\nabla x_{1n} \times y_n)^\top \\ (\nabla x_{2n} \times y_n)^\top \\ (\nabla x_{3n} \times y_n)^\top \end{bmatrix} \\ &= \begin{bmatrix} (\nabla x_{11} \times y_1)^\top \\ (\nabla x_{22} \times y_2)^\top \\ (\nabla x_{33} \times y_3)^\top \end{bmatrix} + \begin{bmatrix} (\nabla x_{12} \times y_2)^\top + (\nabla x_{13} \times y_3)^\top \\ (\nabla x_{23} \times y_3)^\top \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ (\nabla x_{21} \times y_1)^\top \\ (\nabla x_{31} \times y_1)^\top + (\nabla x_{32} \times y_2)^\top \end{bmatrix} \\ &= - \begin{bmatrix} (\text{axl}^{-1} y_1 \nabla x_{11})^\top \\ (\text{axl}^{-1} y_2 \nabla x_{22})^\top \\ (\text{axl}^{-1} y_3 \nabla x_{33})^\top \end{bmatrix} - \begin{bmatrix} (\text{axl}^{-1} y_2 \nabla x_{12})^\top + (\text{axl}^{-1} y_3 \nabla x_{13})^\top \\ (\text{axl}^{-1} y_3 \nabla x_{23})^\top \\ 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 \\ (\text{axl}^{-1} y_1 \nabla x_{21})^\top \\ (\text{axl}^{-1} y_1 \nabla x_{31})^\top + (\text{axl}^{-1} y_2 \nabla x_{32})^\top \end{bmatrix} \\ &= - \text{mat} \begin{bmatrix} \text{axl}^{-1} y_1 & 0 & 0 \\ 0 & \text{axl}^{-1} y_2 & 0 \\ 0 & 0 & \text{axl}^{-1} y_3 \end{bmatrix} \begin{bmatrix} \nabla x_{11} \\ \nabla x_{22} \\ \nabla x_{33} \end{bmatrix} \\ &\quad - \text{mat} \begin{bmatrix} 0 & \text{axl}^{-1} y_3 & -\text{axl}^{-1} y_2 \\ -\text{axl}^{-1} y_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\nabla x_{23} \\ \nabla x_{13} \\ -\nabla x_{12} \end{bmatrix} \\ &\quad - \text{mat} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \text{axl}^{-1} y_1 \\ \text{axl}^{-1} y_2 & -\text{axl}^{-1} y_1 & 0 \end{bmatrix} \begin{bmatrix} \nabla x_{32} \\ -\nabla x_{31} \\ \nabla x_{21} \end{bmatrix} \\ &= \text{mat}(L_{\text{diag}, Y} \hat{\nabla} \text{diagvec } X + L_{\text{skew}, Y} \hat{\nabla} \text{skewvec } X + L_{\text{sym}, Y} \hat{\nabla} \text{symvec } X) \end{aligned}$$

## References

- [1] S. Anicic, H. Le Dret, and A. Raoult, *The infinitesimal rigid displacement lemma in Lipschitz co-ordinates and application to shells with minimal regularity*, Math. Meth. Appl. Sci. **27** (2004), no. 11, 1283–1299.

[2] P.G. Ciarlet, *Mathematical Elasticity, Vol. III: Theory of Shells.*, first ed., North-Holland, Amsterdam, 1999.

[3] ———, *On Korn's inequality.*, Chin. Ann. Math. **31B** (2010), no. 5, 607–618.

[4] P.G. Ciarlet and C. Mardare, *On rigid and infinitesimal rigid displacements in three-dimensional elasticity*, Math. Models Methods Appl. Sci. **13** (2003), no. 11, 1589–1598. MR 2024464 (2004j:74014)

[5] A. Klawonn, P. Neff, O. Rheinbach, and S. Vanis, *FETI-DP domain decomposition methods for elasticity with structural changes: P-elasticity*, ESAIM: Math. Mod. Num. Anal. **45** (2011), 563–602.

[6] A. Korn, *Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen.*, Bulletin International de l'Académie des Sciences de Cracovie Classe des Sciences Mathématiques et Naturelles (1909), no. 9, Novembre, 705–724.

[7] P. Neff, *On Korn's first inequality with nonconstant coefficients.*, Proc. Roy. Soc. Edinb. A **132** (2002), 221–243.

[8] ———, *Local existence and uniqueness for a geometrically exact membrane-plate with viscoelastic transverse shear resistance.*, Math. Meth. Appl. Sci. (MMAS), DOI 10.1002/mma.597 **28** (2005), 1031–1060.

[9] ———, *Local existence and uniqueness for quasistatic finite plasticity with grain boundary relaxation.*, Quart. Appl. Math. **63** (2005), 88–116.

[10] ———, *Existence of minimizers for a finite-strain micromorphic elastic solid.*, Proc. Roy. Soc. Edinb. A **136** (2006), 997–1012.

[11] ———, *A geometrically exact planar Cosserat shell-model with microstructure. Existence of minimizers for zero Cosserat couple modulus.*, Math. Mod. Meth. Appl. Sci.(M3AS) **17** (2007), no. 3, 363–392.

[12] P. Neff and I. Münch, *Curl bounds Grad on SO(3).*, ESAIM: Control, Optimisation and Calculus of Variations, DOI 10.1051/cocv:2007050 **14** (2008), no. 1, 148–159.

[13] P. Neff, D. Pauly, and K.-J. Witsch, *Maxwell meets Korn: A new coercive inequality for tensor fields in  $\mathbb{R}^{N \times N}$  with square-integrable exterior derivative*, Math. Methods Appl. Sci. **35** (2012), 65–71.

[14] P. Neff, D. Pauly, and K.J. Witsch, *A canonical extension of Korn's first inequality to  $H(\text{Curl})$  motivated by gradient plasticity with plastic spin*, C. R. Acad. Sci. Paris, Ser. I **349** (2011), 1251–1254.

[15] ———, *Poincaré meets Korn via Maxwell: Extending Korn's first inequality to incompatible tensor fields.*, submitted; <http://arxiv.org/abs/1203.2744>; Preprint SM-E-753, Universität Duisburg-Essen, Schriftenreihe der Fakultät für Mathematik, <http://www.uni-due.de/~hn213me/preprints/bd753.pdf> (2012).

[16] P. Neff and W. Pompe, *Counterexamples in the theory of coerciveness for linear elliptic systems related to generalizations of Korn's second inequality.*, submitted (2011).

[17] W. Pompe, *Korn's first inequality with variable coefficients and its generalizations.*, Comment. Math. Univ. Carolinae **44,1** (2003), 57–70.

[18] ———, *Counterexamples to Korn's inequality with non-constant rotation coefficients.*, Math. Mech. Solids **16** (2011), 172–176, doi: 10.1177/1081286510367554.

[19] W.P. Ziemer, *Weakly Differentiable Functions.*, Graduate Texts in Mathematics, vol. 120, Springer, Berlin, 1989.